ISRAEL JOURNAL OF MATHEMATICS, Vol. 48, No. 4, 1984

THE DIMENSION CONJECTURE FOR POLYDISC ALGEBRAS

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ABSTRACT
It is shown that for $m \neq n$ the polydisc algebras $A(D^m)$ and $A(D^n)$ are not isomorphic as Banach spaces. More precisely, there is no linear embedding of the dual space $A(D^n)^*$ into $A(D^m)^*$ for $m < n$. The invariant is infinite dimensional and is based on certain multi-indexed martingales related to those considered by Davis et al. [10]. In the one-dimensional case, i.e. for the space $A(D)^*$, a finite inequality is proved, implying that $A(D^2)^*$ is not finitely representable in $A(D)^*$. Extensions to algebras on products of strictly pseudo-convex domains are outlined. They imply in particular the non-isomorphism of certain algebras in the same number of variables, for instance $A(D^4) \not\approx A(B_2 \times B_2)$.

0. Introduction

Denote $D = \{z \in \mathbf{C} : |z| < 1\}$ the open disc and Π the unit circle (group) equipped with Haar measure. For $d = 1, 2, \ldots$, let $A(D^d)$ be the m-polydisc algebra, thus the algebra of complex valued continuous functions on $\bar{D}^d$ which are analytic on D^d. If $A(D^d)$ is equipped with the sup-norm, it becomes a Banach space. The restriction map $A(D^d) \to C(\Pi^d) : f \mapsto f|_{\partial D^d}$ is an isometric embedding, identifying $A(D^d)$ with the translation invariant space on Π^d of those continuous functions f such that

$$\hat{f}(k_1 \ldots, k_d) = \int f(\theta_1, \ldots, \theta_d) \exp(-i(k_1\theta_1 + \cdots + k_d\theta_d)) d\theta_1 \ldots d\theta_d$$

vanishes whenever $\inf_{1 \leq j \leq d} k_j < 0$. In the sequel, polydisc algebras will be seen as function algebras on the torus.

The subject of this paper is the isomorphic classification of the Banach spaces $A(D^d)$. The main result, solving the so-called dimension conjecture (see [18], section 11) for polydisc algebras, is

Received March 13, 1984

THEOREM 1. *For $m \neq n$, there is no linear isomorphism between $A(D^m)$ and $A(D^n)$.*

Partial solutions were obtained by G. M. Henkin [15], B. S. Mitjagin and A. Pelczynski [18] and recently the author in [3]. In [3], the non-isomorphism of $A(D^2)$ and $A(D^3)$ was shown, using an analysis of the dual space $A(D^2)^*$. This space can be proved to be isomorphic with the space

$$A(D)^* \hat{\otimes} A(D)^*,$$

a result used in the argument. It was in fact more generally shown (see [3]) that

PROPOSITION 1.

$$A(D^d)^* \simeq A(D)^* \hat{\otimes} \cdots \hat{\otimes} A(D)^* \quad (d\text{-fold})$$
$$\simeq X_d \oplus (X_{d-1} \hat{\otimes} M)$$

where $M = M(\Pi)$ is the usual measure space and

$$X_d = L^1(\Pi)/_{H_0^1} \hat{\otimes} \cdots \hat{\otimes} L^1(\Pi)/_{H_0^1} \quad (d\text{-fold}),$$
$$H_0^1 = \{f \in L^1(\Pi)\,;\, \hat{f}(n) = 0 \text{ for } n < 0\}.$$

Its knowledge can however be omitted in proving Theorem 1. More precisely, it will be shown that

THEOREM 1'. *For $m < n$, there is no linear embedding of $A(D^n)^*$ into $A(D^m)^*$.*

One may write

$$\text{dual } C^*\text{-algebra} \not\supseteq A(D)^* \not\supseteq A(D^2)^* \not\supseteq \cdots \not\supseteq A(D^\infty)^*$$

where $A(D^\infty)$ is the analytic subspace of $C(\Pi^\infty)$, Π^∞ the infinite torus. It is indeed known that each space $A(D^d)^*$ is weakly complete, while $A(D^\infty)^*$ is not, since the characters

$$e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_k}, \ldots$$

form a complemented l^1-sequence in $A(D^\infty)$ (see [6]).

The analogue of Theorem 1 for H^1-spaces, thus the non-isomorphism of the Banach spaces $H^1(\Pi^m)$ and $H^1(\Pi^n)$ for $m \neq n$, was obtained earlier (see [4] and [5]). The results stated as Theorem 1 and Theorem 1' are again of infinite dimensional nature, and we don't know at this point how to localize them, except in two cases listed below as Theorem 2 and Theorem 3.

Let X, Y be Banach spaces. Say that X is finitely representable (f.r.) in Y provided there is a constant $\lambda < \infty$ and for each finite dimensional subspace E of X a subspace F of Y of the same dimension, such that $d(E,F) \leqq \lambda$, where

$$d(E,F) = \inf\{\|T\|\,\|T^{-1}\|;\ T: E \to F \text{ linear onto-isomorphism}\}$$

is the Banach–Mazur distance.

THEOREM 2 (G. Pisier). *$A(D)^*$ is not f.r. in the dual of a C^*-algebra.*

THEOREM 3. *$A(D^2)^*$ is not f.r. in $A(D)^*$.*

The invariant from which Theorem 1′, Theorem 2 and Theorem 3 are derived is the behaviour of certain multi-indexed martingales ranging in the space we consider.

Let X be a Banach space. A d-indexed martingale $F = F(\theta^1, \dots, \theta^d)$ on $\Omega = \Omega_1 \times \cdots \times \Omega_d$ ($\Omega_j = \Pi^\infty$) will be called a complex martingale of type (d) provided it is of the form

$$F = \sum_{k_1,\dots,k_d=1}^{\infty} A_{k_1,\dots,k_d}(\theta_1^1, \dots, \theta_{k_1-1}^1, \dots, \theta_1^d, \dots, \theta_{k_d-1}^d) \exp(i(\theta_{k_1}^1 + \cdots + \theta_{k_d}^d))$$

where A_K, $K = (k_1, \dots, k_d)$, is an X-valued function.

Define also

$$\rho(F) = \inf_K \left\| \int_\Omega A_K \right\|_X.$$

Say that F is uniformly bounded provided

$$\|F\|_{L_X^\infty} = \sup_{k_1,\dots,k_d \in \mathbf{Z}_+} \|\mathbf{E}_{k_1,\dots,k_d}[F]\|_{L_X^\infty} < \infty$$

where $\mathbf{E}_{k_1,\dots,k_d} = \mathbf{E}_{k_1} \otimes \cdots \otimes \mathbf{E}_{k_d}$ are the natural product expectations on Ω.

Let first $X = A(D)^*$ and F the X-valued type-(1) martingale (considered by G. Pisier)

$$F(\theta) = \sum_{k=1}^{\infty} A_k(\theta_1, \dots, \theta_{k-1}) \exp(i\theta_k)$$

where for each $k = 1, 2, \dots$

$$A_k(\theta) = \frac{1}{2} \prod_{l=1}^{k-1} \left(1 + \frac{1}{2}\exp(i\theta_l)\exp(-i2^l\xi) + \frac{1}{2}\exp(-i\theta_l)\exp(i2^l\xi)\right) \exp(-i2^k\xi),$$

considered as an element of $L^1(\xi \in \Pi)/H_0^1(\xi)$. Since clearly

$$\mathbf{E}_k[F](\theta)=\prod_{l=1}^{k}(1+\cos(\theta_l-2^l\xi))-1,$$

both members considered again as elements of $L^1(\xi\in\Pi)/H_0^1(\xi)$, it follows that $\|\mathbf{E}_k[F]\|_{L_X^\infty}\leqq 2$. Also, for each k

$$\int_\Omega A_k=\tfrac{1}{2}\exp(-i2^k\xi)$$

implying $\rho(F)=\frac{1}{2}$.

By d-fold tensoring $F\otimes\cdots\otimes F$ a uniformly bounded $A(D^d)^*$-valued complex martingale of type (d) is found for which $\rho(F)>0$. Theorems (1′), (2) and (3) are therefore consequences of the following results.

THEOREM 1″. *Let F be a uniformly bounded $A(D^d)^*$-valued martingale of type $(d+1)$. Then $\rho(F)=0$.*

THEOREM 2′. *Let X be the dual of a C^*-algebra and F an X-valued martingale of type* (1). *Then*

$$\sqrt{k}\,\rho(F)\leqq C\|\mathbf{E}_k[F]\|_{L_X^1},\qquad\forall k$$

where C is a numerical constant.

THEOREM 3′ *Let X be the space L^1/H_0^1. If F is an X-valued martingale of type* (2), *then*

$$\sqrt{k}\,\rho(F)\leqq C\|\mathbf{E}_{k,k}[F]\|_{L_X^1},\qquad\forall k$$

where again C is a numerical constant.

Theorem 2′ follows by an iteration argument from the inequality (cf. [10], Th.8) due to U. Haagerup (cf. [11], [12])

$$\frac{1}{2\pi}\int_0^{2\pi}\|a+e^{i\theta}b\|\,d\theta\geqq(\|a\|^2+\delta^2\|b\|^2)^{1/2}$$

(δ = numerical constant), valid for a, b in the dual of a C^*-algebra.

Theorem 3′ will be shown in the next section and relies on the same methods used to prove the cotype 2 property of L^2/H_0^1.

The proof of Theorem 1″ is given in section 2 of this paper. We proceed by induction on d. The reasoning relies on sequence arguments and does not provide an analogue of Theorems 2′ and 3′. In particular, it does not imply the non-isomorphism of the H^∞-spaces $H^\infty(D^2)$ and $H^\infty(D^3)$. At this point, the local

Banach space theory of the algebras of analytic functions in several variables is essentially non-existent.

A classical invariant to distinguish $H^\infty(D)$ and $H^\infty(D^2)$ is the so-called $(i_p-\pi_p)$ ratio defined for a Banach space X by

$$k_p(X) = \sup\{i_p(u) \mid u \text{ linear operator on } X \text{ and } \pi_p(u) \leqq 1\}.$$

We don't explicitly define the p-summing (resp. p-integral) norm $\pi_p(u)$ (resp. $(i_p(u))$ here, since they will not be used. The reader not familiar with those notions may consult [18].

Now if $X = H^\infty(D)$, then

$$k_p(X) \leqq C\frac{p^2}{p-1} \qquad (1 < p < \infty)$$

while $k_p(X) \geqq c_p^m$ in case $X = H^\infty(D^m)$ and $2 < p < \infty$ (see again [18]). Very recently, it appeared that in fact $k_p(X) = \infty$ whenever $p \neq 2$ for $X = H^\infty(D^m)$ and $m > 1$ ([2]).

The same phenomenon happens for H^∞-spaces on complex balls in more than 1-variable (see [7]). Hence the latter invariant becomes useless in a multi-variable context.

Let us recall some further notions for later use. Define $\mathcal{H}$ as the subspace of those functions $h \in L^1(\Pi^\infty)$ such that the natural martingale difference sequence of $h = h(\psi_1, \psi_2, \ldots)$, i.e.,

$$\Delta_k[h] = (\mathbf{E}_k - \mathbf{E}_{k-1})[h] \qquad (k = 1, 2, \ldots),$$

has the property that for each k the difference $\Delta_k[h]$ is an H_0^1-function w.r.t. the variable ψ_k. The formal projection on $\mathcal{H}$ is given by $\frac{1}{2}(\mathrm{Id} + i\mathbf{H})$ where

$$\mathbf{H} = H_{\psi_1}\mathbf{E}_1 + H_{\psi_2}\mathbf{E}_2 + \cdots + H_{\psi_k}\mathbf{E}_k + \cdots$$

and H is the usual Hilbert-transform acting on $L^1(\Pi)$. Thus $\mathbf{H}$ maps real functions on real functions and behaves like a martingale transform.

Its regularity properties follow indeed from those of H using the transference formula

$$\mathbf{H} = \lim_{k\to\infty}\left\{\lim_{n_k\to\infty}\cdots\lim_{n_1\to\infty}(S^k_{n_k\theta}\cdots S^1_{n_1\theta})^{-1}H_\theta(S^k_{n_k\theta}\cdots S^1_{n_1\theta})\right\}$$

where $S^k_{n_k\theta}$ is the measure preserving transformation of Π^∞ obtained by translation $\psi_k \mapsto \psi_k + n_k\theta$.

In proving Theorem 1″, we use a Rudin–Shapiro construction to generate bounded analytic functions. Let us recall the method. Let (λ_k) be a scalar sequence such that $\Sigma |\lambda_k|^2 < \infty$ and (γ_k) a uniformly bounded sequence of functions. Define by induction the pair (α_k, β_k) as follows:

$$\alpha_1 = \lambda_1 \gamma_1, \qquad \beta_1 = 1,$$

$$\begin{cases} \alpha_{k+1} = \alpha_k + \lambda_{k+1} \bar{\beta}_k \gamma_{k+1}, \\ \beta_{k+1} = \beta_k - \lambda_{k+1} \bar{\alpha}_k \gamma_{k+1}. \end{cases}$$

Then clearly

$$|\alpha_{k+1}|^2 + |\beta_{k+1}|^2 = (1 + |\lambda_{k+1}\gamma_{k+1}|^2)(|\alpha_k|^2 + |\beta_k|^2)$$

and by iteration

$$|\alpha_k|^2 + |\beta_k|^2 \leqq \prod_{j=1}^{\infty} (1 + |\lambda_j|^2 \|\gamma_j\|_\infty^2)$$

yielding in particular that (α_k) is a uniformly bounded sequence of functions.

1. Proof of Theorem 3′

Let $X = L^1/H_0^1$ and $F = \Sigma_{j,k=1}^n F_{j,k}(\theta_1, \dots, \theta_{j-1}, \psi_1, \dots, \psi_{k-1}) \exp(i(\theta_j + \psi_k))$ a complex X-valued martingale of type (2). Theorem 3′ is a consequence of the next inequality to be shown:

$$(1) \qquad \left(\sum_{j=1}^{n} \left\| \int F_{j,n-j}(\theta, \psi) d\theta d\psi \right\|^2 \right)^{1/2} \leqq C \| F \|_{L_X^1},$$

where C is a numerical constant. Notice that (1) clearly implies

$$(2) \qquad \left(\sum_{j,k=1}^{n} \left\| \int F_{j,k} \right\|^2 \right)^{1/2} \leqq C \sqrt{n} \| F \|_{L_X^1}.$$

The verification of (1) is based on the same technique as used to prove the cotype 2 property of X. Let us denote by C various numerical constants. The main ingredient is the following result (see [2], Theorem 1.1):

PROPOSITION 2. *Given $\Delta \in L_+^1(\Pi)$, $\int \Delta = 1$, there exist $\Delta_1 \geqq \Delta$, $\int \Delta_1 \leqq C$ and a projection P from $L^2(\Delta_1)$ onto $H_0^2(\Delta_1)$ which is $L^p(\Delta_1)$-$L^p(\Delta_1)$ bounded for $1 < p < \infty$ and $L^1(\Delta_1)$-$L^1(\Delta_1)$-weak bounded.*

The notation $H^p(\mu)$ stands for the closure of the analytic trigonometric polynomials in the space $L^p(\mu)$, μ = Radon probability measure on Π. The projection P generalizes the classical Riesz projection operator. More variable analogues of Proposition 2, even in a restricted sense, are known to be untrue (see [2], [7]). This fact is related to the failure of the $(i_p - \pi_p)$-property mentioned in the introduction.

The following fact follows from H^p-theory or from [10] (see Theorem 1).

LEMMA 3. *If* $0 < p < \infty$ *and*

$$f(\theta, \psi) = \sum f_{j,k}(\theta_1, \dots, \theta_{j-1}, \psi_1, \dots, \psi_{k-1}) \exp(i(\theta_j + \psi_k))$$

is a **C**-*valued martingale of type* (2), *then*

$$\left(\sum_{j,k} \| f_{j,k} \|_{L^p}^2 \right)^{1/2} \leqq C_p \| f \|_p.$$

Define, with F as above,

$$x_j = \int F_{j,n-j}(\theta, \psi) d\theta d\psi \qquad (1 \leqq j \leqq n).$$

Denote $q : L^1(\Pi) \to X$ the quotient-map and consider functions $f_j \in L^1(\Pi)$ with $q(f_j) = x_j$ and such that

$$I = \int \left(\sum |f_j|^2 \right)^{1/2}$$

is minimum up to factor 2, the minimum taken over all sequences $f_1, \dots, f_n$ of liftings of $x_1, \dots, x_n$. It will be shown that

$$I \leqq C \| F \|_{L^1_X} \tag{3}$$

obviously implying (1).

Define $\Delta(\xi) = I^{-1}(\Sigma |f_j(\xi)|^2)^{1/2}$ and apply Proposition 2 yielding $\Delta_1 \geqq \Delta$ and the projection P. Of course, we may suppose $\Delta_1 \geqq 1$ pointwise on Π and consequently consider an outer function ϕ with $|\phi| = \Delta_1$ on $\partial D = \Pi$.

Let G be the convolution on $\Pi^\infty \times \Pi^\infty$ of F and the scalar function

$$\alpha(\theta, \psi) = 2 \prod_{j=1}^{n} (1 + \cos(\theta_j + \psi_{n-j})).$$

Thus

$$\| G \|_{L^1_X} \leqq 2 \| F \|_{L^1_X} \tag{4}$$

and $G(\theta, \psi) = \Sigma_{j,k} G_{j,k}(\theta_1, \dots, \theta_{j-1}, \psi_1, \dots, \psi_{k-1}) \exp(i(\theta_j + \psi_k))$, where clearly $G_{jk} = 0$ for $j + k < n$ and $G_{j,n-j}$ is the constant function x_j.

Fix $0 < p < 1$ and define $Q = \mathrm{Id} - P$, vanishing on analytic trigonometric polynomials of mean zero. For fixed θ, $\psi \in \Pi^\infty$, we may write (with slight abuse of notation)

$$\| G(\theta,\psi)\|_X = \inf_{h\in H_0^1} \int | G(\theta,\psi)(\xi) - h(\xi)|\, d\xi$$

$$= \inf_{h\in H_0^1} \int | G(\theta,\psi)(\xi)\phi(\xi)^{-1} - h(\xi)\phi(\xi)^{-1} | \Delta_1(\xi) d\xi$$

$$\geqq C_p \| Q[G(\theta,\psi)\phi^{-1} \|_{L^p(\Delta_1)}$$

since Q satisfies Kolmogorov's theorem w.r.t. the density Δ_1. Integration in θ, ψ yields by (4)

$$\left\{ \int \left| \sum_{j,k} Q[G_{j,k}(\theta,\psi)\phi^{-1}](\xi)\exp(i(\theta_j+\psi_k))\right|^p \Delta_1(\xi) d\theta d\psi d\xi \right\}^{1/p} \leqq C_p \| F\|_{L^1_X}.$$

Now for ξ fixed, the left member is the L^p-norm of a scalar martingale of type (2), to which Lemma 3 applies. Taking the remark on the $G_{j,n-j}$ into account, as well as the fact that $q(f_j) = x_j$ $(1 \leqq j \leqq n)$, we get

$$\left\{ \int \left[\sum_j | Q[f_j\phi^{-1}] |^2 \right]^{p/2} \Delta_1 \right\}^{1/p} \leqq C_p \| F \|_{L^1_X}. \tag{5}$$

The proof is now concluded using a standard extrapolation reasoning based on Hölder's inequality. Take $0 < \tau < 1$ satisfying

$$1 = (1-\tau)2 + \tau p.$$

Since for each j

$$f_j - \phi Q[f_j\phi^{-1}] \in H_0^1$$

it follows from the definition of I that

$$2I \leqq \int \left(\sum_{j=1}^n | \phi Q[f_j\phi^{-1}] |^2 \right)^{1/2}$$

$$= \int \left(\sum | Q[f_j\phi^{-1}] |^2 \right)^{1/2} \Delta_1$$

$$\leqq \left\| \left(\sum_j | Q[f_j\phi^{-1}] |^2 \right)^{1/2} \right\|_{L^2(\Delta_1)}^{(1-\tau)2} \left\| \left(\sum_j | Q[f_j\phi^{-1}] |^2 \right)^{1/2} \right\|_{L^p(\Delta_1)}^{\tau p}.$$

By the $L^2(\Delta_1)$-boundedness of Q and (5), this gives

$$2I \leqq C_p \left\{ \int_\Pi \left(\sum_j | f_j |^2 \right) | \phi |^{-2} \Delta_1 \right\}^{(1-\tau)} \| F \|_{L^1_X}^{\tau p}.$$

Since $|\phi| = \Delta_1$ on Π and $\Delta_1 \geqq I^{-1}(\Sigma_j |f_j|^2)^{1/2}$, it follows that

$$I \leqq C_p I^{1-\tau}\left\{\int\left(\sum_j |f_j|^2\right)^{1/2}\right\}^{1-\tau} \|F\|^{\tau p},$$

$$I^{\tau p} \leqq C_p \|F\|_{L^1_X}^{\tau p},$$

hence (3) and the theorem.

Simple examples show that the minoration in Theorem 3′ is best possible.

REMARK. It is possible to give a more direct proof of Theorem 3′, not relying on Proposition 2. The method is the real-variable approach presented in [9] to show the cotype 2 and Grothendieck property of $A(D)^*$.

2. Proof of Theorem 1″

Let us denote (d) the statement in (1″), for simplicity. Its verification will be done by induction. Statement (1) was obtained in the previous section. Part of the argument, such as the convolution, will reappear below.

It should be said that in fact the proof of $(d-1)$ already implies the non-embeddability of $A(D^{d+1})^*$ into $A(D^d)^*$. This is a consequence of Proposition 1. Indeed, the continuum-direct sum in l^1-sense $\bigoplus_{l^1(c)} X_d$ is a subspace of $A(D^{d+1})^*$ and since

$$A(D^d)^* \simeq X_d \oplus (X_{d-1} \hat{\otimes} M),$$

X_d being a separable Banach space, a simple argument yields the embedding of X_d into $X_{d-1} \hat{\otimes} M$. The latter space would therefore admit a uniformly bounded type (d) martingale F with $\rho(F) > 0$. But the argument proving $(d-1)$ works as well for $X_{d-1} \hat{\otimes} M$, giving a contradiction. In particular Theorems 2′ and 3′ imply Theorem 1 for m, $n \leqq 3$.

The remainder of this section deals with the implication $(d-1) \Rightarrow (d)$. As said, the result is of infinite dimensional nature.

Consider the partial order on complexes $K \in \mathbf{Z}_+^d$ defined by $K < K'$ provided $k_i < k_i'$ $(1 \leqq i \leqq d)$, $K = (k_1, \dots, k_d)$ and $K' = (k_1', \dots, k_d')$.

Denote for each $s = 1, 2, \dots$ by R_s at the 1-variable kernel on Π with trapezoidal Fourier transform such that

$$\hat{R}_s(k) = 1 \qquad \text{for } |k| \leqq s,$$

$$\hat{R}_s(k) = 0 \qquad \text{for } |k| \geqq 2s.$$

Hence $\|R_s\|_1 \leqq 3$.

For fixed dimension d, consider the translation invariant operators

$$P_S = \mathrm{Id} - (\mathrm{Id}^1 - R_s^1)\otimes\cdots\otimes(\mathrm{Id}^d - R_s^d)$$

acting on any invariant space on Π^d. Here I_d^j (resp. R_s^j) denotes the identity operator (resp. the R_s-convolution) w.r.t. the variable $\xi_j \in \Pi$. Clearly $P_s\varphi = 0$ provided

$$\operatorname{Spec}\varphi \subset \prod_{j=1}^{d}\{k \in \mathbf{Z} \mid |k| \geqq 2s\}.$$

On the other hand, it is easily seen that each P_s maps $A(D^d)$ (resp. $A(D^d)^*$) on a subspace isomorphic to $A(D^{d-1})$ (resp. $A(D^{d-1})^*$).

Let

$$F(\theta) = \sum_{k_1,\dots,k_{d+1}=1}^{\infty} A_{k_1,\dots,k_{d+1}}(\theta_1^1,\dots,\theta_{k_1-1}^1,\dots,\theta_1^{d+1},\dots,\theta_{k_{d+1}-1}^{d+1})\exp(i(\theta_{k_1}^1+\cdots+\theta_{k_{d+1}}^{d+1}))$$

be a uniformly bounded $A(D^d)^*$-valued martingale of type $(d+1)$. We will show that $\rho(F)=0$. Let us simplify notation by denoting ψ the Π^∞-variable θ^{d+1} and ω the variable $(\theta^1,\dots,\theta^d)\in\Omega=\Pi^\infty\times\cdots\times\Pi^\infty$.

Rewrite F as

$$F = \sum_{K,k} A_{K,k}(\omega,\psi_1,\dots,\psi_{k-1})\exp(i\omega_K)\exp(i\psi_k)$$

where

$$\omega_K = \sum_{1\leqq j\leqq d}\theta_{k_j}^j, \qquad K=(k_1,\dots,k_d).$$

It is clear that for fixed $k = 1,2,\dots,$ the formula

$$F^{(k)}(\omega) = \sum_K\left(\int A_{K,k}(\omega,\psi)d\psi\right)\exp(i\omega_K)$$

defines a uniformly bounded complex martingale of type (d). Hence by a previous observation and the induction hypothesis, it follows that

$$\rho(P_s(F^{(k)})) = 0, \qquad \forall k, \quad \forall s. \tag{1}$$

Fix some large integer n. Fact (1) allows one to introduce inductively complexes $K_r \in \mathbf{Z}_+^d$,

$$K_n < K_{n-1} < K_{n-2} < \cdots < K_1 \tag{2}$$

such that

$$\| P_{s_k}(x_k) \| \approx 0 \qquad \text{where } x_k = \int A_{K_k,k}(\omega, \psi) d\omega d\psi.$$

In this construction, we let k run from n up to 1. At each step, the choice of the integer s_k depends on the elements $x_n, x_{n-1}, \dots, x_{k+1}$ which were already obtained. More precisely, there is for each $k = 1, \dots, n$ some function $\varphi_k \in A(D^d)$, $\| \varphi_k \| \leqq 1$ and φ_k with finite spectrum satisfying, taking $\varphi'_k = \varphi_k - P_{s_k}(\varphi_k)$,

$$\langle x_k, \varphi'_k \rangle = \tfrac{1}{2} \| x_k \| \geqq \tfrac{1}{2} \rho(F) \tag{3}$$

and s_k will be chosen according to $\varphi_n, \dots, \varphi_{k+1}$. The condition will appear later in the Rudin–Shapiro construction.

Define again G as the convolution on $\Omega \times \Pi^\infty$ of F and α, where

$$\alpha(\omega, \psi) = 2 \prod_{k=1}^{n} [1 + \cos(\omega_{K_k} + \psi_k)].$$

Thus G has the form, using (2),

$$G(\omega, \psi) = \sum_{k=1}^{n} \exp(i\psi_k) \sum_{l=1}^{k} G_{k,l}(\omega_{K_{l+1}}, \dots, \omega_{K_k}, \psi_l, \dots, \psi_{k-1}) \exp(i\omega_{K_l})$$

where

$$G_{k,k} = \int A_{K_k,k}(\omega, \psi) d\omega d\psi = x_k. \tag{4}$$

The reader will easily make the verification.

Defining $\eta = (\eta_1, \dots, \eta_n)$, $\eta_l = \omega_{K_l} \in \Pi$, G can be rewritten as

$$G(\omega, \eta) = \sum_{l=1}^{n} \left\{ \sum_{k=l}^{n} G_{k,l}(\eta_{l+1}, \dots, \eta_k, \psi_l, \dots, \psi_{k-1}) \exp(i\psi_k) \right\} \exp(i\eta_l). \tag{5}$$

Also, letting $X = A(D^d)^*$

$$\| G(\omega, \eta) \|_X \leqq 2 \| F \|_{L^\infty_X}. \tag{6}$$

To minorate the left side, we construct test-functions in $A(D^d)$, depending on the variables ω, η. This is achieved by a backwards ($l = n, n-1, \dots, 1$) Rudin–Shapiro construction, based on the functions φ_k. Define

$$\begin{cases} U_n = \dfrac{1}{\sqrt{n}} \exp(i(\eta_n + \psi_n)) \varphi'_n \\ V_n = 1 \end{cases}$$

and recursively

$$\begin{cases} U_l = U_{l+1} + \dfrac{1}{\sqrt{n}} \exp(i(\eta_l + \psi_l)) Z_{l+1} \bar{V}_{l+1} \varphi'_l \\[2ex] V_l = V_{l+1} - \dfrac{1}{\sqrt{n}} \exp(i(\eta_l + \psi_l)) Z_{l+1} \bar{U}_{l+1} \varphi'_l \end{cases}$$

where $Z_{l+1} = Z_{l+1}(\eta_{l+1} + \psi_{l+1}, \dots, \eta_n + \psi_n)$ remains to be defined. Now Z_{l+1} is uniformly bounded by 1 and depends on the functions $\varphi_n, \dots, \varphi_{l+1}$ only. Thus U_m, V_m $(m > l)$ as well depend only on $\varphi_n, \dots, \varphi_{l+1}$. Therefore the integer s_l considered above can be taken such that the φ'_l-multiplication yields $U_l(\eta, \psi)$, $V_l(\eta, \psi)$ in $A(D^d)$. In fact, we are only interested in U_1.

From their definition, U_l and V_l appear as uniformly bounded complex type (1)-martingales in the variables

$$\eta_n + \psi_n, \eta_{n-1} + \psi_{n-1}, \dots, \eta_l + \psi_l.$$

Using the biorthogonality in η, it follows by (5) and (6) that

$$\begin{aligned} C \| F \|_{L^\infty_X} &\geqq \int \langle G(\eta, \psi), U_1(\eta, \psi) \rangle d\eta d\psi \\ &= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \int \left\langle \sum_{k=l}^{n} G_{k,l} \exp(i\psi_k), \exp(i\psi_l) Z_{l+1} \bar{V}_{l+1} \varphi'_l \right\rangle d\psi. \end{aligned} \tag{7}$$

We will now make Z_{l+1} precise. First, by successive applications of Jensen's inequality

$$\int \log | V_{l+1} | d\psi \geqq \int \log | V_{l+2} | d\psi \geqq \cdots \geqq 0$$

so that $\exp(-\int \log | V_{l+1} | d\psi)$ defines a 1-bounded function on Π^d. Take

$$Z_{l+1} = \frac{V_{l+1}}{| V_{l+1} |} \exp\left(- \int \log | V_{l+1} | d\psi \right) \exp(-i\mathbf{H}_\psi (\log | V_{l+1} |)) \tag{8}$$

where $\mathbf{H}$ is the transformation related to the space $\mathscr{H}$ considered in the introduction. Clearly $\| Z_{l+1}(\eta, \psi) \|_\infty \leqq 1$ and a standard perturbation argument allows one to assume the spectrum (w.r.t. $\xi \in \Pi^d$) contained in some finite subset of $\mathbf{Z}^d$. The latter property holding for U_{l+1}, V_{l+1} and since φ'_l has finite spectrum, it will still be satisfied by U_l, V_l.

From (8)

$$Z_{l+1} \bar{V}_{l+1} = \exp\left(- \int \log | V_{l+1} | d\psi \right) \exp[\log | V_{l+1} | - i\mathbf{H}_\psi (\log | V_{l+1} |)]$$

showing that scalarly, thus $(Z_{l+1} \bar{V}_{l+1})(\eta, \psi)(\xi)$ for fixed $\xi \in \Pi^d$, one gets an element of

$$\mathscr{H}(-(\eta_{l+1} + \psi_{l+1}), \dots, -(\eta_n + \psi_n)).$$

Also, by construction

$$\int Z_{l+1}\bar{V}_{l+1}d\psi = 1.$$

Therefore, the lth term in the sum (7) equals

$$\int\left\langle \sum_{k=l}^{n} G_{k,l}\exp(i\psi_k), \exp(i\psi_l)\varphi'_l\right\rangle d\psi = \langle x_l, \varphi'_l\rangle = \tfrac{1}{2}\| x_l \|$$

by (3) and (4). We proved that

$$C\| F\| \geqq \frac{\sqrt{n}}{2}\rho(F).$$

Since n was arbitrary, $\rho(F)$ equals zero, completing the proof.

3. Remarks and extensions of the method

(1) The proof of Theorem 1″ actually shows that there is no linear embedding of the $(d+1)$-fold projective tensor product

$$L^1(\Pi)\hat{\otimes} L^1/H_0^1 \underbrace{\hat{\otimes}\cdots\hat{\otimes}}_{d\text{ times}} L^1H_0^1$$

in the space $A(D^d)^*$. Let us mention that the non-isomorphism of the spaces $A(D)$ and $A(D)\hat{\hat{\otimes}} C(\Pi)$ was known to A. Pelczynski. Also, by the results of P. Wojtaszczyk (see [20])

$$l^1(\mathbf{N})\hat{\otimes} L^1/H_0^1 \simeq L^1/H_0^1.$$

(2) The notion of type (d)-complex martingale is closely tied up with the group-structure of the torus Π^d. It may be that related ideas permit one to solve the dimension conjecture for the ball algebras $A(B_m)$. It is known that $A(D)\not\simeq A(B_m)$ and, from [7], also $H^\infty(D)\not\simeq H^\infty(B_m)$ for $m>1$.

(3) Assume U a domain of holomorphy in $\mathbf{C}^m$, $m = m_1+\cdots+m_d$, obtained as a product $U = U_1\times\cdots\times U_d$, where each U_j is a strictly pseudoconvex bounded closed domain in $\mathbf{C}^{m_j}$ with C^2-smooth boundary. Let $V = V_1\times\cdots\times V_{d'}$ be another such domain in $\mathbf{C}^{m'}$.

The argument presented in the previous section permits one to prove that the algebras $A(U)$ and $A(V)$ are not linearly isomorphic whenever $d\neq d'$. This result provides the non-isomorphism of many pairs of algebras of analytic functions in the same number of complex variables.

In particular $A(D^m)\not\simeq A(B_m)$ if $m>1$, a theorem due to G. M. Henkin [14].

Also $A(D^4) \neq A(B_2 \times B_2)$, a result not covered by Henkin's method. In what follows, we briefly outline the proof of the statement. We restrict ourselves to $U_j = B_{m_j}$, the unit ball in $\mathbf{C}^{m_j}$. The generalization to products of arbitrary strictly pseudoconvex domains is based on the theory developed in [15]. A detailed exposition of the case $U = B_m$ can be found in [8].

By a recent result of A. B. Aleksandrov on the existence of inner functions [1], there exists $\varphi \in H^\infty(B_m)$ such that $\varphi(0) = 0$ and $|\varphi| = 1$ almost everywhere on ∂B_m. The existence of inner functions for strictly pseudoconvex domains was proved by E. Löw [16]. Clearly the map

$$\sum_{k \in \mathbf{Z}}{}' a_k e^{ik\theta} \mapsto \sum a_k \varphi^k$$

yields an isometric embedding of $L^1(\Pi)/H^1_0$ into $(A(B_m))^*$. Consequently, if $U = \Pi_{1 \leq j \leq d} B_{m_j}$, there exists a uniformly bounded complex martingale F of type (d) ranging in $A(U)^*$ and satisfying $\rho(F) > 0$. Therefore it remains to show that if F is some uniformly bounded $A(U)^*$-valued martingale of type $(d+1)$, necessarily $\rho(F) = 0$. This is shown by induction on d. The argument for $d = 1$ is essentially the same as to show the implication $(d) \Rightarrow (d+1)$. It again uses the Rudin–Shapiro construction and is a straightforward adaptation of the polydisc-algebra proof. The reproducing kernel of $A(B_m)$ is the Szego kernel

$$K(z, \zeta) = (1 - \langle z, \zeta \rangle)^{-m}.$$

We denote P_r $(0 \leqq r \leqq 1)$ the Poisson kernel, i.e.

$$P_r f(z) = \tilde{f}(rz) \qquad \text{for } f \in L^1(\partial B_m) \quad \text{and} \quad z \in \partial B_m,$$

where $\tilde{f}$ is the harmonic extension of f.

Consider the operators $(0 \leqq r < 1)$

$$Q_r = \mathrm{Id} - (\mathrm{Id}^1 - P_r^1) \otimes \cdots \otimes (\mathrm{Id}^d - P_r^d)$$

acting on $A(U)$ or on $A(U)^*$. Then, again by induction hypothesis, it is not hard to see that

$$\rho(Q_r F^{(k)}) = 0, \qquad \forall k = 1, 2, \ldots \quad \text{and} \quad r < 1.$$

Thus in order to generate $A(U)$ members with the Rudin–Shapiro construction it suffices to verify the property

$$(*) \qquad \operatorname{dist}(\tau(\mathrm{Id} - Q_r)\varphi, A(U)) \to 0 \qquad \text{if } r \to 1$$

uniformly for φ in the unit-ball of $A(U)$, for a fixed continuous function τ on

∂U. We may and will assume τ of the form $\tau(\xi)=\tau_1(\zeta_1)\dots\tau_d(\zeta_d)$ where τ_j is a polynomial in $z_1,\dots,z_{m_j},\bar z_1,\dots,\bar z_{m_j}$ for $j=1,\dots,d$.

At this point the strict pseudoconvexity of the domains $U_j=B_{m_j}$ is exploited. The method to show (*) is well-known. The reproducing kernel K of $A(U)$ is the product $K^1\otimes\cdots\otimes K^d$. Denote $[L,\kappa]$ the commutator defined by

$$[L,\kappa](f)=L(\kappa . f)-\kappa L(f).$$

We estimate

$$(**)\qquad \|\tau(\mathrm{Id}-Q_r)\varphi-K[\tau(\mathrm{Id}-Q_r)\varphi]\|_\infty.$$

Now (**) equals $\|[K,\tau](\mathrm{Id}-Q_r)\varphi\|_\infty$ and

$$[K,\tau]=[K^1\otimes\cdots\otimes K^d,\tau_1\otimes\cdots\otimes\tau_d]=\sum_{S\subsetneq\{1,\dots,d\}}\prod_{j\in S}\tau_j.\bigotimes_{j\in S}K_j\otimes\bigotimes_{j\notin S}[K_j,\tau_j].$$

Again since $(\mathrm{Id}-Q_r)\varphi$ is in $A(U)$, (**) tends to zero for $r\to 1$ since if $S\subset\{1,\dots,d\}$, $S\neq\varnothing$

$$\lim_{\rho\lesssim 1}\left\|\bigotimes_{j\in S}[K_j,\tau_j](\mathrm{Id}^j-P_r^j)\varphi\right\|_\infty=0.$$

Indeed, for each j, $f\in L^\infty(\partial B_{m_j})$ and $z\in\partial B_{m_j}$

$$[K_j,\tau_j]f(z)=\lim_{\rho\lesssim 1}\int_{\partial B_{m_j}}\frac{\tau_j(\zeta)-\tau_j(z)}{(1-\langle\rho z,\zeta\rangle)^{m_j}}f(\zeta)\sigma(d\zeta),$$

$$[K_j,\tau_j](f-P_rf)(z)=\lim_{r\lesssim 1}\int_{\partial B_{m_j}}(\mathrm{Id}-P_r)_\zeta\left\{\frac{\tau_j(\zeta)-\tau_j(z)}{(1-\langle\rho z,\zeta\rangle)^{m_j}}\right\}f(\zeta)\sigma(d\zeta),$$

and

$$\left\{\frac{\tau_j(\circ)-\tau_j(z)}{(1-\langle\rho z,\circ\rangle)^{m_j}}\,\middle|\,0\leqq\rho<1,\ z\in\partial B_{m_j}\right\}$$

is a compact subset of $L^1(\partial B_{m_j})$. The reader will find details in [19], chapter 9.

Knowing (*), it is easy to repeat the argument of the previous section.

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